

Entry Task: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for

$$z = 5\ln(x) - \underbrace{x^4}_{F} \underbrace{y^2 e^{3x}}_S - \frac{4}{y^2}$$

$$\frac{\partial z}{\partial x} = 5 \cdot \frac{1}{x} - 3x^4 y^2 e^{3x} - 4x^3 y^2 e^{3x} - 0$$

$$F = x^4$$

$$S = y^2 e^{3x}$$

$$F' = 4x^3$$

$$S' = y^2 e^{3x} \cdot 3$$

$$z = 5\ln(x) + x^4 y^2 e^{3x} - 4y^{-2}$$

$$\frac{\partial z}{\partial y} = 0 + x^4 \cdot 2y e^{3x} + 8y^{-3}$$

$$= 2x^4 y e^{3x} + \frac{8}{y^3}$$

Interpreting as a rate

Your company produces and sells **two** products (hats and sunglasses)

x = number of hats

y = number of glasses

You find that profit is given by

$$P(x, y) = -3x^2 + 30x - 5y^2 + 130y + 2xy - 100$$

1. Find the partial derivatives.

2. Find and interpret

$$P_x(5, 8) \text{ and } P_y(5, 8).$$

$$P_x = -6x + 30 + 2y$$

$$P_y = -10y + 130 + 2x$$

$$P_x(5, 8) = -6(5) + 30 + 2(8) = 16 = \frac{\partial P}{\partial x} \leftarrow \begin{array}{l} \text{dollar in profit per} \\ \text{hats} \end{array}$$

"The sale of the next hat will increase profit by about \$16." \uparrow after (5, 8)

$$P_y(5, 8) = -10(8) + 130 + 2(5) = -80 + 130 + 10 = 60 = \frac{\partial P}{\partial y} \leftarrow \begin{array}{l} \text{dollar in profit per} \\ \text{sunglasses} \end{array}$$

"The sale of the next sunglasses will increase profit by about \$60."

3. Estimate the values of

$$\frac{P(5.001, 8) - P(5, 8)}{0.001} \approx P_x(5, 8) = 16$$

$$\frac{P(5, 8.01) - P(5, 8)}{0.01} \approx P_y(5, 8) = 60$$

y -changing

x -changing

From HW: (Cobb-Douglas Model)

Q = units produced

K = capital expenditures

(in thousand dollars)

L = hours of labor

$$Q = 75K^{1/3}L^{2/3}$$

Assume there are:

\$2,744,000 in capital expenditures and
4913 in total labor hours.

Find and interpret $\frac{\partial Q}{\partial K}$ and $\frac{\partial Q}{\partial L}$.

$$\frac{\partial Q}{\partial K} = 75 \cdot \frac{1}{3} K^{-2/3} L^{2/3} = 25 K^{-2/3} L^{2/3}$$

$$\frac{\partial Q}{\partial L} = 75 K^{1/3} \cdot \frac{2}{3} L^{-1/3} = 50 K^{1/3} L^{-1/3}$$

$$K = 2744 \text{ thousand dollars}$$

$$L = 4913 \text{ hours}$$

$$\frac{\partial Q}{\partial K} = 25 (2744)^{-2/3} (4913)^{2/3}$$

$$\approx 36.8622$$

$$\approx 37 \text{ units/thousand dollars}$$

INCREASE CAPITAL EXPENDITURES
BY 1 thousand dollars
leads to an increase in units produced
of about 37.

$$\frac{\partial Q}{\partial L} = 50 (2744)^{1/3} (4913)^{-1/3}$$

$$\approx 41.1765$$

$$\approx 41 \text{ units/labor hour}$$

INCREASE LABOR HOURS BY 1

⇒ INCREASE IN UNITS BY 41.

Definition

A point (a,b) is a **critical point** for a function $z = f(x,y)$ if BOTH

$$f_x(a,b) = 0 \quad \text{and} \quad f_y(a,b) = 0.$$

Going back to our last example:

$$P(x,y) = -3x^2 + 30x - 5y^2 + 130y + 2xy - 100$$

Find the critical point.

$$P_x = -6x + 30 + 2y \stackrel{?}{=} 0$$

\Rightarrow

$$2y = 6x - 30$$

\Rightarrow

$$y = 3x - 15$$

AND

$$P_y = -10y + 130 + 2x \stackrel{?}{=} 0$$

$$-10(3x - 15) + 130 + 2x \stackrel{?}{=} 0$$

$$\Rightarrow -30x + 150 + 130 + 2x \stackrel{?}{=} 0$$

$$\Rightarrow 280 - 28x = 0$$

$$280 = 28x$$

$$\boxed{x = 10}$$

$$y = 3x - 15$$

$$y = 3(10) - 15 = 15$$

ONLY CRITICAL

POINT $(x,y) = (10, 15)$

Check!!!

ASIDE

MAX PROFIT

$$= P(10, 15)$$

$$= \dots = 1025$$

Graphical Interpretation

Pretend you are skiing on the surface

$$z = f(x, y) = 15 - x^2 - y^2$$

1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

2. Find and interpret $f_x(7,4)$ and $f_y(7,4)$

3. Find the critical point.

$$\frac{\partial z}{\partial x} = -2x \Rightarrow \frac{\partial z}{\partial x} = -14 \approx$$

$$\frac{\partial z}{\partial y} = -2y \Rightarrow \frac{\partial z}{\partial y} = -8 \approx$$

$$\frac{f(7.0001, 4) - f(7, 4)}{0.0001} = \text{slope}$$

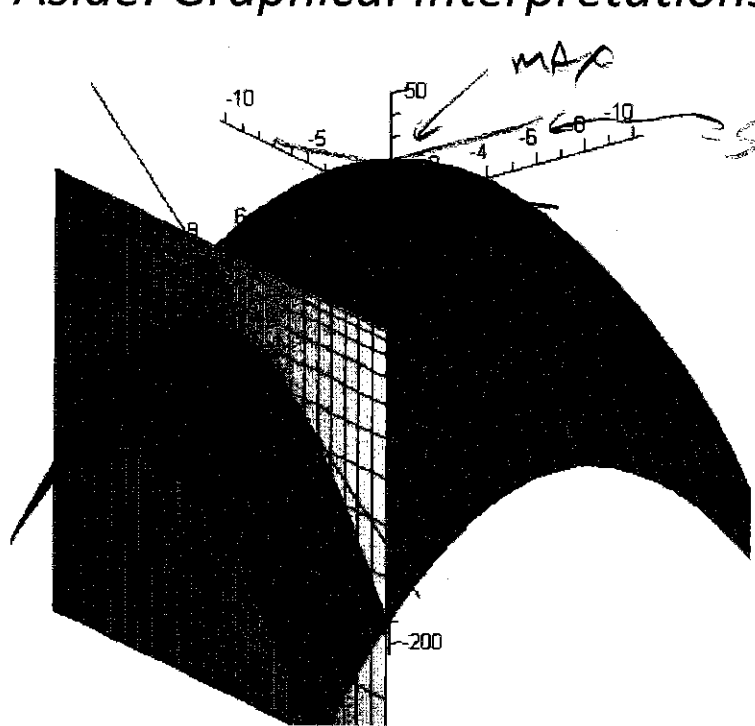
$$\frac{f(7, 4.00001) - f(7, 4)}{0.00001} = \text{slope}$$

$$-2x = 0 \quad \text{And} \quad -2y = 0$$

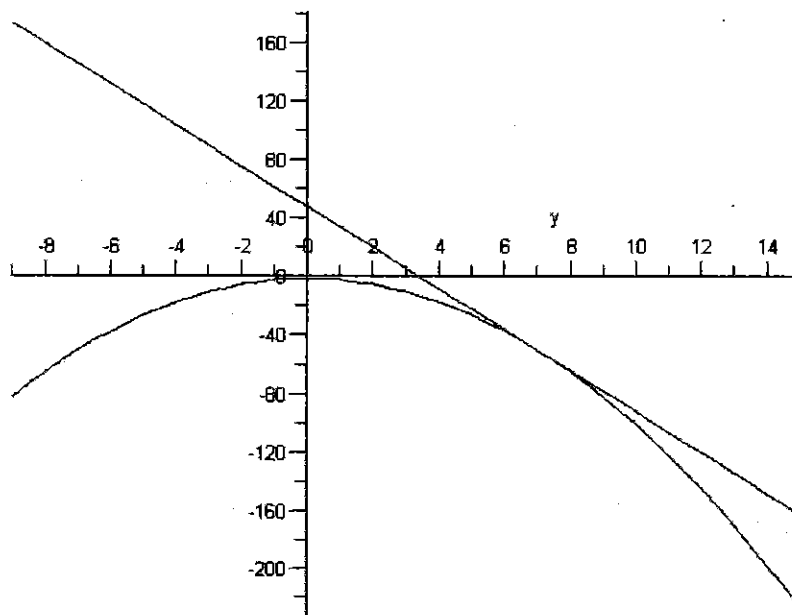
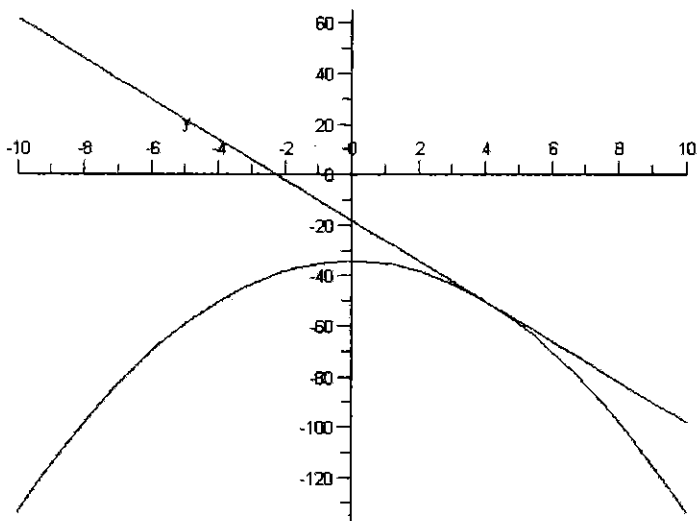
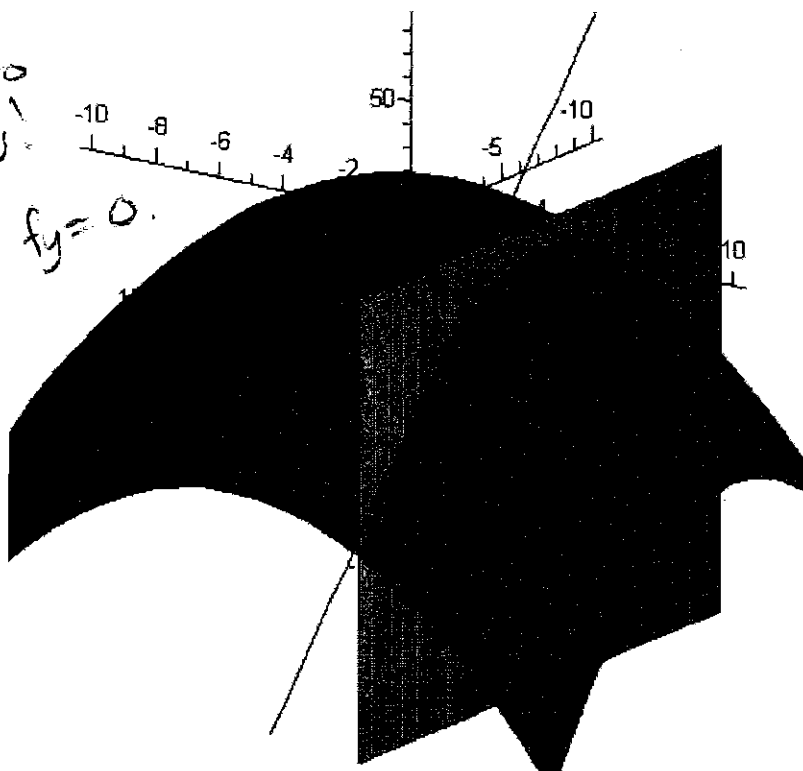
$$\Rightarrow x = 0 \quad \text{Ans} \quad y = 0$$

$$(0, 0)$$
$$f(0, 0) = 15 \leftarrow \text{MAX}$$

Aside: Graphical Interpretations



slope is zero
both directions!
AT TOP
 $f_x = 0$ AND $f_y = 0$.



Example:

Find all critical points of

$$f(x, y) = 2x^4 + y^2 - 4xy + 1$$

$$f_x = 8x^3 - 4y \stackrel{?}{=} 0$$

AND

$$f_y = 2y - 4x \stackrel{?}{=} 0$$

$$\Rightarrow 8x^3 = 4y$$

$$\Rightarrow y = 2x^3$$

$$2(2x^3) - 4x$$

$$4x^3 - 4x$$

$$4x(x^2 - 1)$$

$$x = 0$$

or

$$x^2 - 1 = 0$$

$$x = -1$$

$$x = 1$$

3 possibilities

$$x = 0 \Rightarrow y = 2(0)^2 = 0 \Rightarrow$$

$$x = -1 \Rightarrow y = 2(-1)^3 = -2 \Rightarrow$$

$$x = 1 \Rightarrow y = 2(1)^2 = 2 \Rightarrow$$

$$(x, y) = (0, 0)$$

$$(x, y) = (-1, -2)$$

$$(x, y) = (1, 2)$$

Critical!